Nonlinear Multi-Scale Finite Element Modeling of the Progressive Collapse of Reinforced Concrete Structures

Lecture 4: Geometrical nonlinearities

Péter Z. Berke

Visiting Professor
Departamento de Engenharia Metalúrgica e de Materiais
Universidade Federal do Ceará, Bloco 729
Scientific Collaborator
Building, Architecture and Town Planning Dept. (BATir)
Université Libre de Bruxelles (ULB), Brussels, Belgium

Inspired and adapted from the ‘Nonlinear Modeling of Structures’ course of Prof. Thierry J. Massart at the ULB

Principle of virtual work

$$\delta W_{int} = \delta W_{ext}$$

with

$$\delta W_{int} = \int_{vol} \text{(stress)} : \delta\text{(conjugate strain)} d(vol)$$

$$\delta W_{ext} = \int_V \vec{f}.\delta\vec{u} dV + \int_S \vec{p}.\delta\vec{u} dS$$

Choice to be made of the description configuration

Choice of configuration (Initial or deformed volume?)

Choice of the strain measure (which tensor?)

Choice of the stress measure (which tensor?)

Sources of nonlinearity

Non proportionality between applied forces and displacements

u^h

nodal displacements

→ **Geometrical non linearity** → Strain measure

- Finite strains
- Large displacements
- Change in boundary conditions as a result of loading

Strain measure → **Material Non linearity** → Stress measure

$$\underbrace{\int_V [B]^t \left\{\sigma^h\left(u^h\right)\right\} dV}_{\{f_{int}\}}$$ ← Stress measure

Objective (nonlinear) strain measures

Stress measures

Principle of virtual work

Principle of virtual work

$$\delta W_{int} = \delta W_{ext}$$

with

$$\delta W_{int} = \int_{vol} (\text{stress}) : \delta(\text{conjugate strain}) d(vol)$$

$$\delta W_{ext} = \int_V \vec{f}.\delta\vec{u}dV + \int_S \vec{p}.\delta\vec{u}dS$$

Choice to be made of the description configuration

Choice of configuration (Initial or deformed volume?)

Choice of the strain measure (which tensor?)

Choice of the stress measure (which tensor?)

Strain measures - uniaxial

Uniaxial stretch

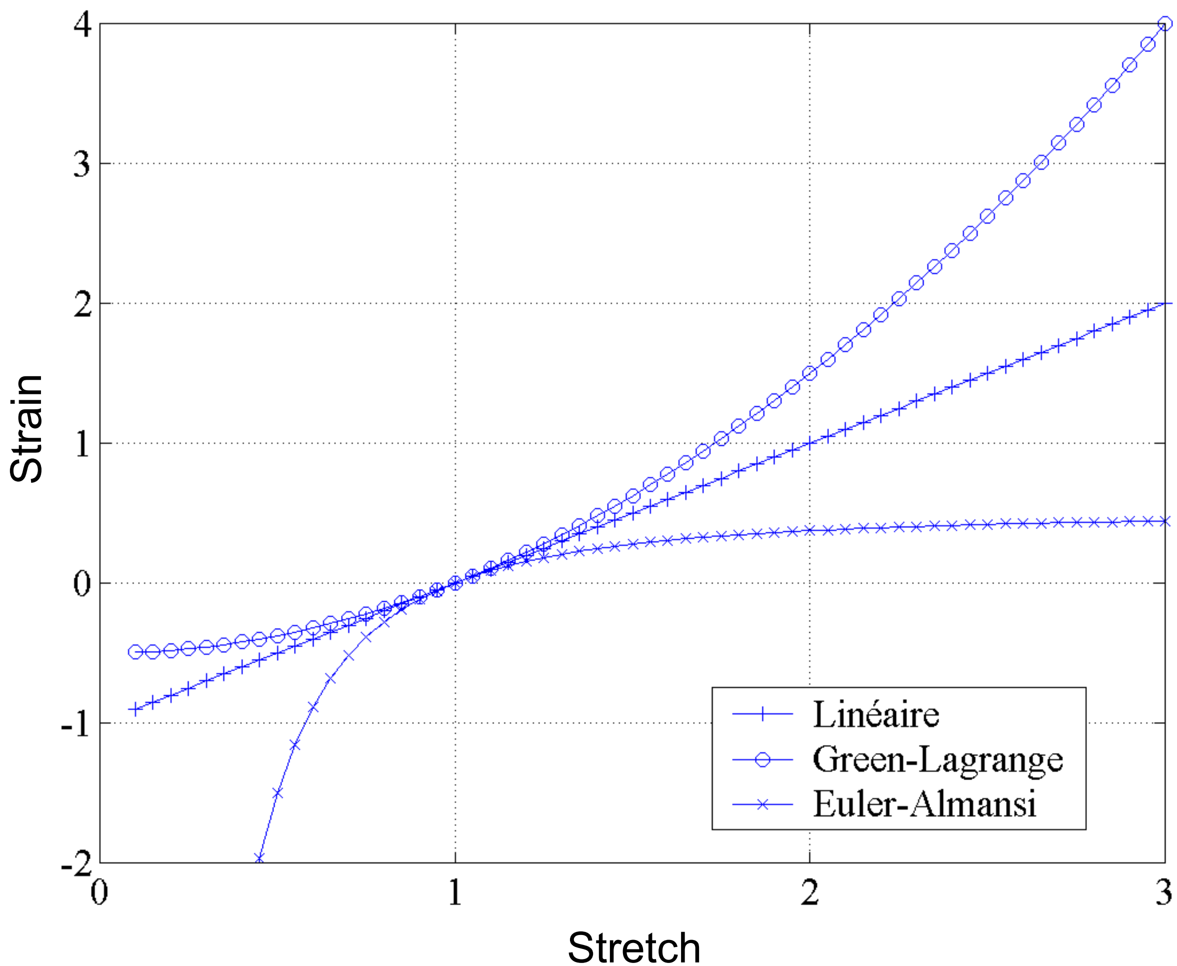
$$\lambda = \frac{l}{l_0}$$

A function $\varepsilon = f(\lambda)$ a 'good' strain measure if

- *f* should vanish when stretch is equal to unity $f(\lambda = 1) = 0$
- *f* is a monotonic increasing function
- *f* is differentiable function
- *f* is such that $f(\lambda)\,|_{\lambda \simeq 1} = \lambda - 1$

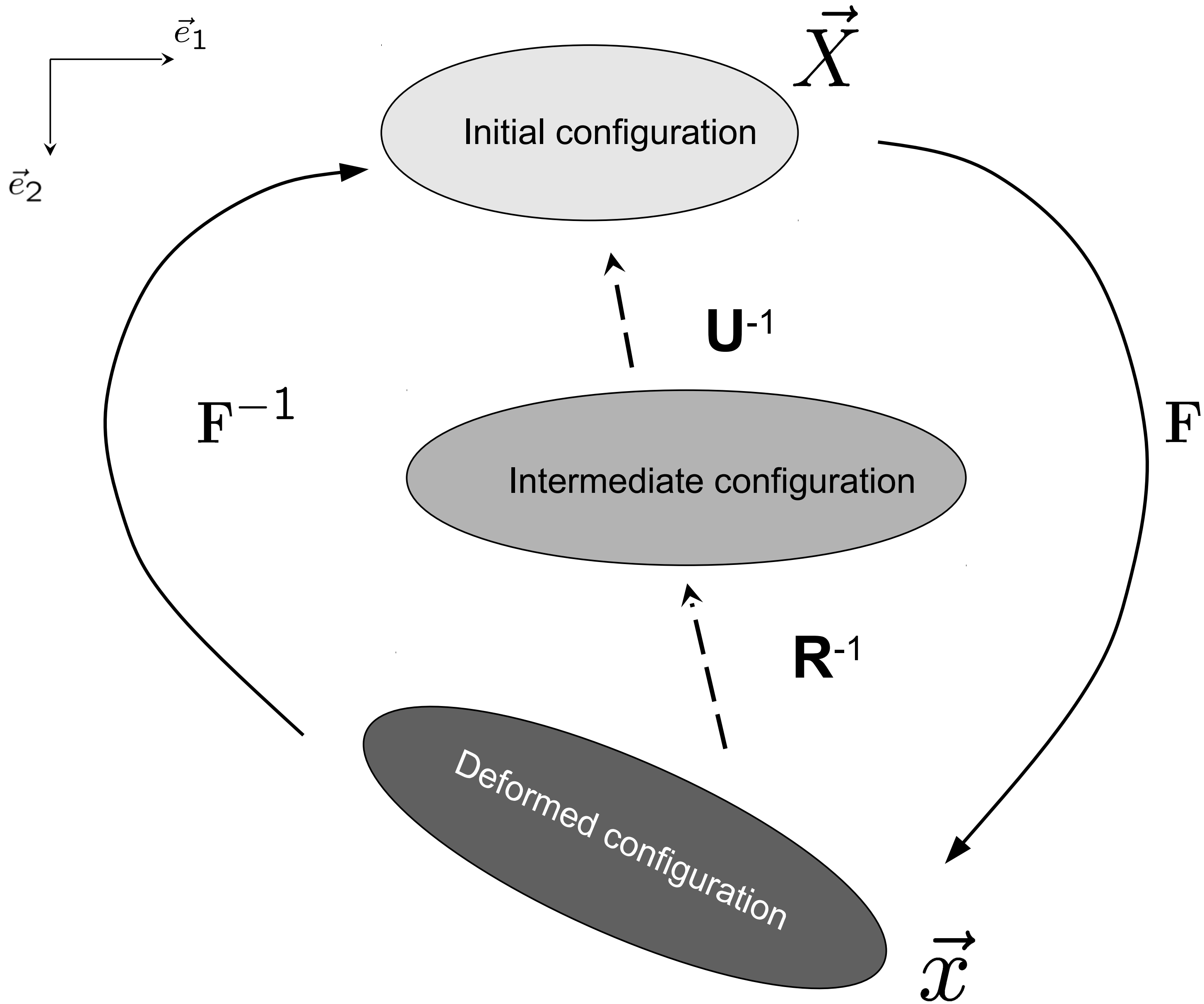
Strain measures - uniaxial

$$\varepsilon_{lineaire} = \lambda - 1 \qquad \varepsilon_{Green} = \frac{1}{2}\left(\lambda^2 - 1\right) \qquad \varepsilon_{Euler} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$$



UNIVERSIDADE FEDERAL DO CEARÁ

F = RU



Transformation relationships

Jacobian of the transformation

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \quad \text{or} \quad \mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}}$$

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} = \frac{\partial X_i + u_i}{\partial X_J} = \delta_{iJ} + \frac{\partial u_i}{\partial X_J}$$

This is often called 'Deformation gradient'

This quantity defines the transformation of an infinitesimal vector

$$dx_i = F_{iJ} dX_J = \frac{\partial x_i}{\partial X_J} dX_J \quad \text{or} \quad d\vec{x} = \mathbf{F} \cdot d\vec{X}$$

Transformation relationships

Transformation of an infinitesimal surface element (Nanson)

$$da = J\mathbf{F}^{-T} dA$$

$$\vec{n} da = J\mathbf{F}^{-T} \cdot \vec{N} dA$$

Transformation of an infinitesimal volume

$$dv = J dV = det(\mathbf{F}) dV$$

Multiaxial strain measures

The quantity $\mathbf{F}$ contains the complete transformation (including the rigid body rotation)

A strain measure should not be rotation sensitive

Polar decomposition of the transformation jacobian

$$\mathbf{F} = \mathbf{R}\ \mathbf{U} \quad \text{with} \quad \begin{aligned} &\mathbf{R}\ \mathbf{R}^T = \mathbf{I} \\ &\mathbf{U} = \mathbf{U}^T = \sqrt{\mathbf{F}\,\mathbf{F}^T} \end{aligned}$$

$\mathbf{R}$ Represents the rigid body rotation

$\mathbf{U}$ Represents the 3D stretch (same role as λ en 1D)

$\rightarrow$ It allows objective strain tensors to be defined

Multiaxial strain measures

Different tensors can be defined

- Biot strain $\mathbf{E}^B = \mathbf{U} - \mathbf{I}$
- Logarithmic strain $\mathbf{E}^N = \ln \mathbf{U}$
- Euler strain tensor $\mathbf{E}^E = \frac{1}{2}\left(\mathbf{I} - \mathbf{U}^{-2}\right)$
- Green strain tensor $\mathbf{E}^G = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T\,\mathbf{F} - \mathbf{I}\right)$

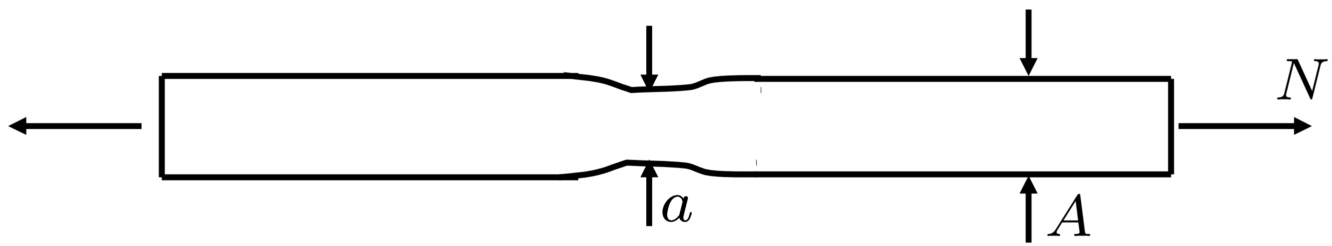
The infinitesimal strain tensor is NOT objective!

$$\cancel{\varepsilon_{jk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_j} + \frac{\partial u_j}{\partial X_k}\right)}$$

Stress measures - uniaxial

The stress measure can also be dependent on the configuration !

Example: Consider a tensile test with necking

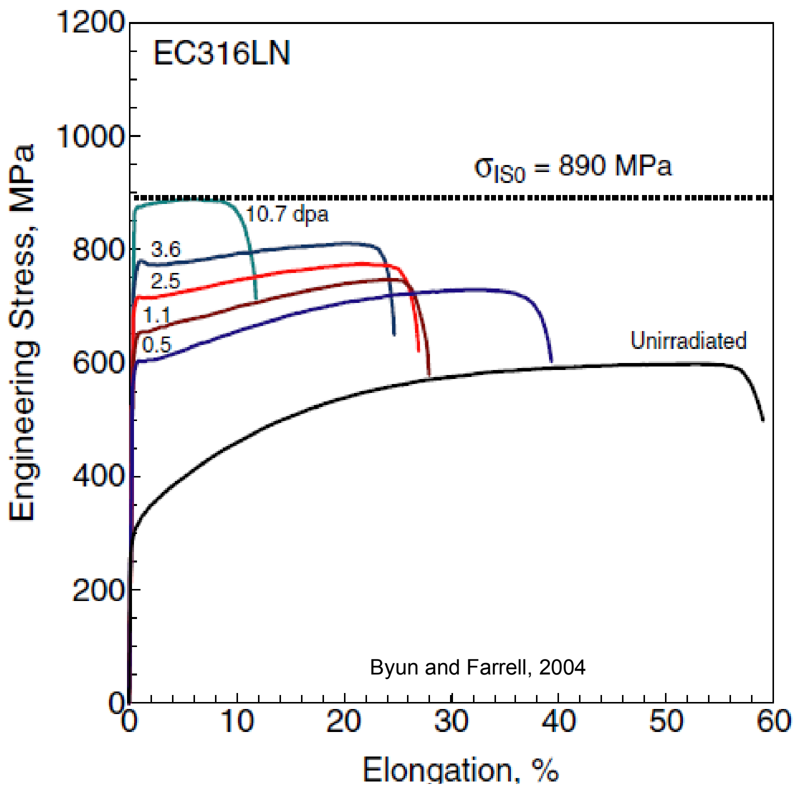


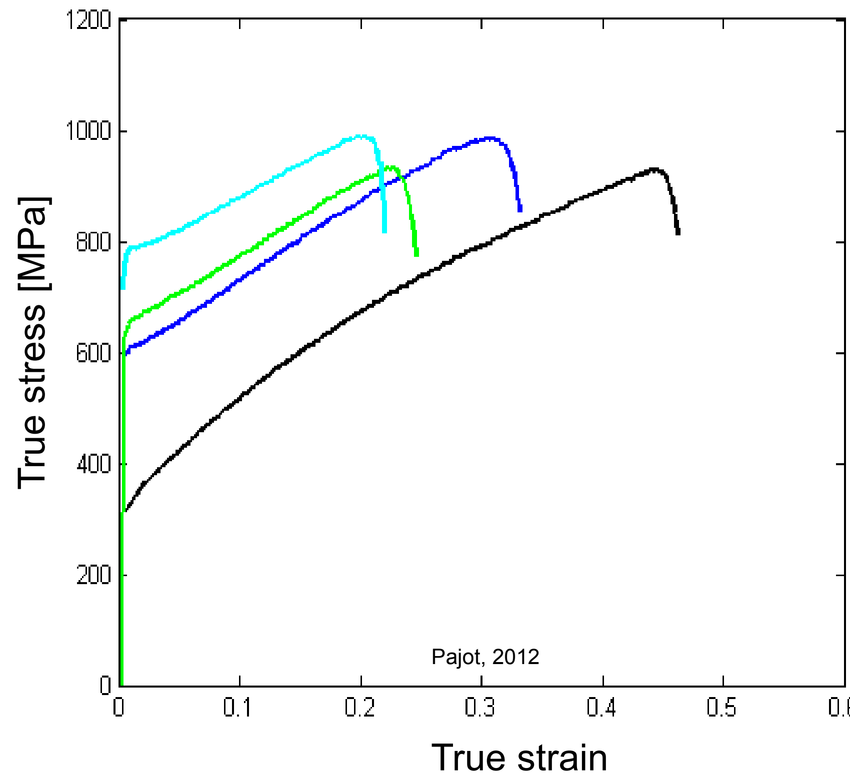
- Initial configuration: $\sigma = \dfrac{N}{A}$ Nominal stress

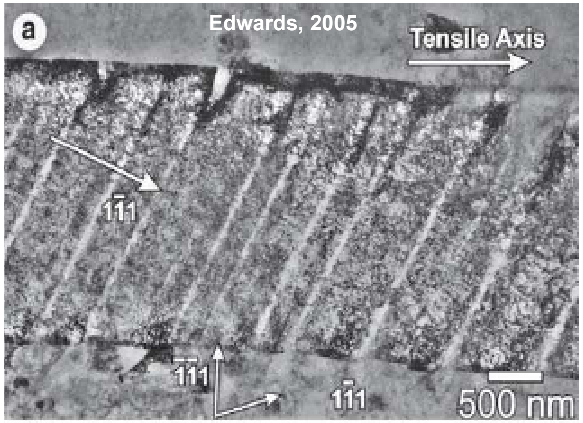
- Deformed configuration $\sigma = \dfrac{N}{a}$ True stress

$\rightarrow$ One should take into account the configuration used in the description in the definition of the stress measure !!!

Stress and strain measures - uniaxial







Stress vectors

True stress vector $$\vec{t} = \lim_{\Delta a \to 0} \frac{\Delta \vec{f}}{\Delta a}$$

'Nominal' stress vector $\vec{T}$

True stress vector relative to the initial area

$$\vec{t} da = \vec{T} dA \rightarrow \vec{T} = \vec{t} \frac{da}{dA}$$

Back transformed nominal stress vector $\vec{T'}$

The deformation gradient allows to transform from the initial to the deformed configuration

$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \quad \text{or} \quad \mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}}$$ **non symmetric !**

Back transformed nominal stress vector $$\vec{T'} = \mathbf{F}^{-1} . \vec{T}$$

Cauchy stress

True force vector per unit DEFORMED area

(~ generalisation of the uniaxial true stress)

Using the definition of the stress tensor σ_{ij}

$$\vec{t}^{(i)} = \sigma_{ij}\vec{e}^{\,j}$$

Expressing the stress vector as a function of the tensor components

$$\vec{t} = n_i \sigma_{ij} \vec{e}^{\,j} \qquad \text{or} \qquad \vec{t} = \boldsymbol{\sigma}.\vec{n}$$

It is related to areas normal in the deformed configuration

It is symmetric (both indices related to the deformed system)

All components non zero even in uniaxial tension if large displacements

Piola-Kirchhoff 1 stress

Defined with respect to initial normals

Per unit initial (undeformed) area

(~generalisation of 1D nominal stress)

Through a relation similar to the Cauchy stress definition

$$\vec{T} = P_{iJ} N_J \vec{e}^{\,i} \qquad \text{or} \qquad \vec{T} = \mathbf{P}.\vec{N}$$

where $\vec{N}$ is the normal vector to the facet in the initial configuration

Relates forces in the deformed configuration to initial non deformed normal areas

NOT symmetric

Difficult to interprete physically

Its transposed is sometimes called the nominal stress tensor

Piola-Kirchhoff 2 stress

Symmetric generalisation of the 1D nominal stress

Based on the back transformed nominal stress vector

Decomposed similarly to the previous tensors

$$\vec{T}' = \mathbf{F}^{-1}.\vec{T} = N_I S_{IJ} \vec{e}^{\,j}$$

Related to initial non deformed normal areas

Symmetric (thanks to the use of $\mathbf{F}^{-1}$)

Difficult to interprete physically

Relationships between stress tensors

Definitions

$$\vec{n}\cdot\boldsymbol{\sigma}da = d\vec{f} = \vec{t}da$$

$$\vec{N}\cdot\mathbf{P}^T dA = d\vec{f} = \vec{T}dA$$

$$\vec{N}\cdot\mathbf{S}dA = \mathbf{F}^{-1}\cdot d\vec{f} = \mathbf{F}^{-1}\cdot\vec{T}dA$$

Nanson's relation

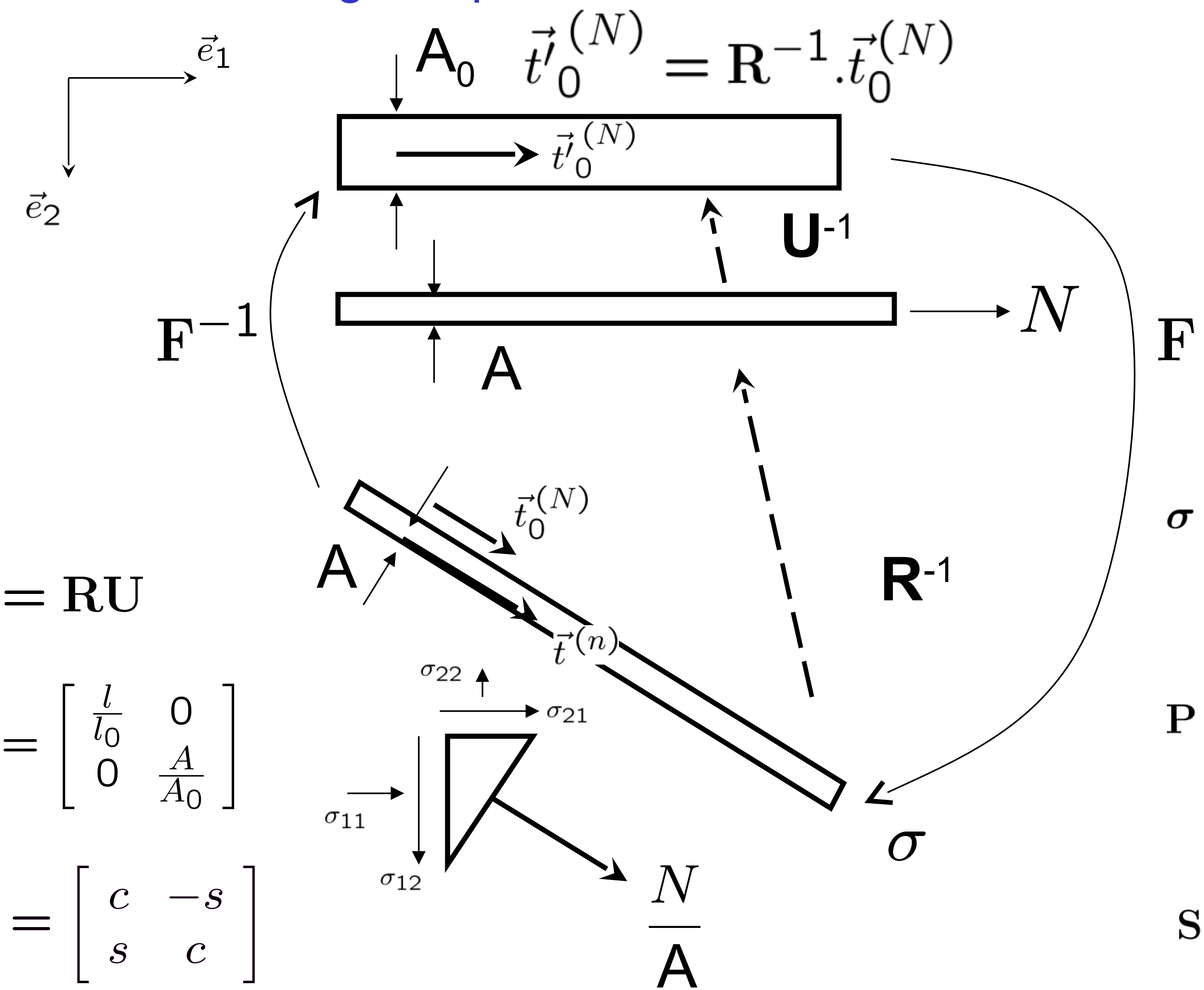
$$\vec{n}da = J\vec{N}\cdot\mathbf{F}^{-1}dA$$

$$\Rightarrow \begin{cases} \mathbf{P}^T & = & J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma} \\ \mathbf{S} & = & \mathbf{P}^T\cdot\mathbf{F}^{-T} & = & \mathbf{F}^{-1}\cdot\mathbf{P} \\ \boldsymbol{\sigma} & = & J^{-1}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T \end{cases}$$

with $J = det(\mathbf{F})$

Interpretation of stress tensors

In the case of large displacements



$$\mathbf{F} = \mathbf{R}\mathbf{U}$$

$$\mathbf{U} = \begin{bmatrix} \frac{l}{l_0} & 0 \\ 0 & \frac{A}{A_0} \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$\boldsymbol{\sigma} = \frac{N}{A}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$$\mathbf{P} = \frac{N}{A_0}\begin{bmatrix} c & 0 \\ s & 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \frac{N}{A_0}\frac{l_0}{l} & 0 \\ 0 & 0 \end{bmatrix}$$

Remarks

The only ‘physical’ is the Cauchy stress !

In the sense that this is the only one that can be compared to a material strength limit

The Cauchy and PK2 stresses are symmetric

The PK1 tensor is non symmetric !

As this is the case for all tensors which have a ‘leg’ (an index) in each configuration (initial and deformed)

The different stress tensors CANNOT be arbitrarily associated to Any strain tensor to represent properly the internal work !

In the case of material nonlinearity an objective stress rate has to be used

Principle of virtual work

$$\delta W_{int} = \delta W_{ext}$$

with

$$\delta W_{int} = \int_{vol} \text{(stress)} : \delta\text{(conjugate strain)}\, d(vol)$$

$$\delta W_{ext} = \int_V \vec{f}.\delta\vec{u} dV + \int_S \vec{p}.\delta\vec{u} dS$$

Conjugate quantities

By definition, two stress and strain quantities are conjugate if their internal product integrated on the proper configuration gives the correct internal work

Deformed configuration

$$\delta W_{int} = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, d(v) = \int_v \sigma_{ij}\delta\varepsilon_{ij}\, d(v)$$

with $$\delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\delta u_i}{\partial x_j} + \frac{\partial\delta u_j}{\partial x_i}\right)$$

Initial configuration

$$\delta W_{int} = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, d(v) = \int_V (J\boldsymbol{\sigma}) : \delta\boldsymbol{\varepsilon}\, d(V)$$

with $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ called the Kirchhoff stress

Initial configuration

$$\delta W_{int} = \int_V \mathbf{P} : \delta \mathbf{g} \, d(V) = \int_V P_{ij} \delta g_{ij} \, d(V)$$

with $$\delta g_{ij} = \frac{\partial \delta u_i}{\partial X_j} = \delta \left(\frac{\partial u_i}{\partial X_j} \right)$$

$$\delta \mathbf{g} = \delta \mathbf{F}$$

Initial configuration

$$\delta W_{int} = \int_V \mathbf{S} : \delta \mathbf{E}\, d(V) = \int_V S_{ij} \delta E_{ij}\, d(V)$$

$$\delta E_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} + \frac{\partial \delta u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial \delta u_k}{\partial X_j}\right)$$

which is the Green strain

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

$$\mathbf{E}^G = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right)$$

Internal work expressions

3 expressions of the internal work variations are thus available

$$w = \int_v \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} \; dv$$

$$w = \int_V \mathbf{P} : \delta\mathbf{F} \; dV$$

$$w = \int_V \mathbf{S} : \delta\mathbf{E} \; dV$$

The expressions of the external work may also depend on the chosen configuration (in case of follower forces – think of a balloon inflating)

UNIVERSIDADE FEDERAL DO CEARÁ

Sources of nonlinearity - summary

Nonlinear nature of the system $\{f_{int}\} = \{f_{ext}\}$

$$\{f_{int}\} = \sum_e \int_{V_e} [B]^T \{\sigma\} \, \mathrm{dV_e}$$

$$\{f_{ext}\} = \sum_e \left(\int_{V_e} [N]^T \{f\} \, \mathrm{dV_e} + \int_{S_{t,e}} [N]^T \left\{ \overline{T}^{(n)} \right\} \mathrm{dS_{t,e}} \right)$$

Geometrical non linearities

Choice of the reference configuration V_e (initial or deformed configuration)

The stress measure should be conjugated to the chosen strain measure

Material non linearities

Non linear relation $\{\sigma\} = f_{nonlinear}(\{\varepsilon\})$

VINCERE TENEBRAS SCIENTIA ULB