Formulation of 1D rate–independent plasticity

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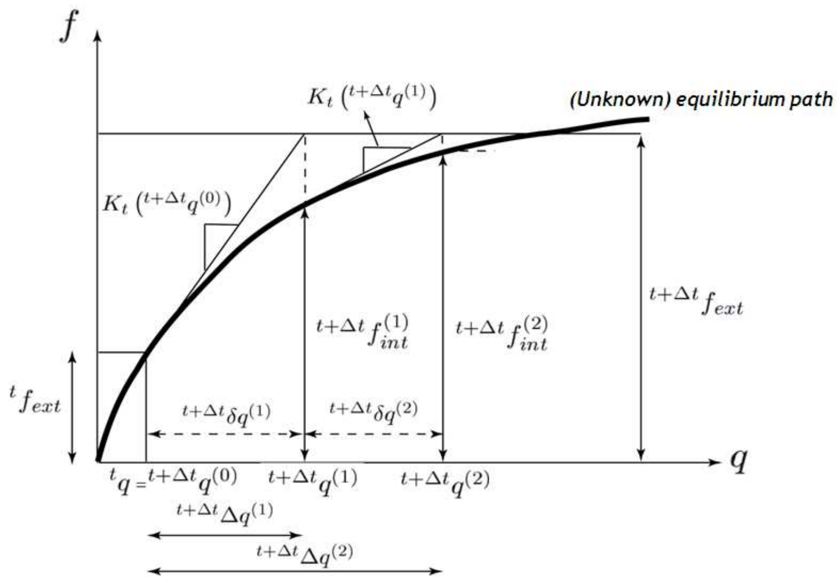

In the following superscripts between parentheses '(i)' denote values at structural iteration i. Variables with $\Delta X^{(i)}$ denote *incremental variation* of the quantities X, while variables with $\delta X^{(i)}$ denote *iterative corrections* of these quantities. Recall that:

$$\Delta X^{(i)} = \sum_{j=1}^{j=i} \delta X^{(j)} \tag{1}$$

1 Problem definition

One of the most developed theories for describing material nonlinearity is the mathematical theory of plasticity. The common idea in all the plasticity models is, that the induced plastic deformation is irreversible. In comparison, the simple elastic behavior can be modeled by a spring: as soon as the loading is removed, the structure returns to the original configuration. In plastic models that are independent of the strain rate (i.e. not viscoplastic) the material response depends only on the loading history. A one dimensional version of such kind of model will be used here to describe the plastic response of a metallic material in a truss structure (made of two–noded bar elements).
In the 1D case there exist only one non zero strain component (the axial strain in a bar) which induces one non zero stress component (the axial stress in a bar).

1.1 Plastic deformation

We shall consider the following equation relating the *incremental variation* in the elastic strain $\Delta\epsilon_e$, the *incremental variation* in the plastic strain $\Delta\epsilon_p$ and the *incremental variation* in the total strain $\Delta\epsilon$:

$$\Delta\epsilon = \Delta\epsilon_e + \Delta\epsilon_p \tag{2}$$

In order to define the limit between the (reversible) elastic and the plastic domain a mathematical function, a so–called *yield function* $f(\sigma^{(k)},\, \kappa^{(k)})$ is used (see following section). The yield function bounds the elastic domain in which

$$\Delta\sigma = \mathrm{E}\,\Delta\epsilon_e \tag{3}$$

sets the dependence of the *incremental variation* of the stress as a function of the *incremental variation* of the elastic strain, with E the elastic modulus of the material. If plastic straining occurs during the increment a part of the *incremental variation* of strain is permanent, or plastic (Eq.(2)). Combining equations (2) with (3) results in:

$$\Delta\sigma = \mathrm{E}\ (\Delta\epsilon - \Delta\epsilon_p) \tag{4}$$

which is the general relationship between the variation of stress and the variation of strain in plasticity.

1.2 Yield function

The yield function $f(\sigma^{(k)}, \kappa^{(k)})$ defines the limit that separates admissible from non–admissible stress states in rate–independent plasticity. It depends on (i) the current stress at state k: $\sigma^{(k)}$ and (ii) the plastic strain history at state k: $\kappa^{(k)}$.
In a one dimensional setting the yield function is defined by:

$$f(\sigma^{(k)}, \kappa^{(k)}) = \mathrm{abs}(\sigma^{(k)}) - \sigma_h(\kappa^{(k)}) \tag{5}$$

with $\sigma_h(\kappa^{(k)})$ the current yield stress of the material, depending on the plastic strain history (see Section 1.4)*. Eq.(5) defines two segments of $\sigma_h(\kappa^{(k)})$ length on the stress axis to the positive and to the negative values from the origin†.

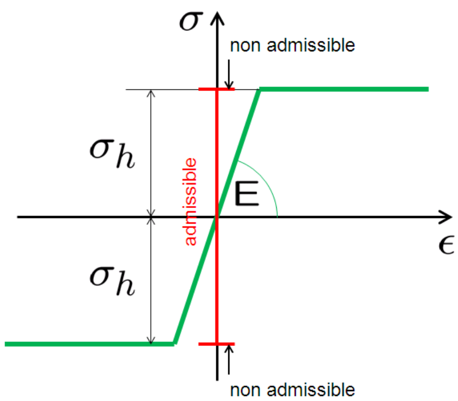

Figure 1: Admissible and non admissible domains defined by the yield function in 1D perfect plasticity, i.e. the yield limit is constant $\sigma_h = \sigma_0$, independent of the plastic strain history.

If the stress at iteration k, $\sigma^{(k)}$, is inside the segment defined by the yield function, the deformation is purely elastic, whereas plastic deformation occurs only if the stress point is on the extremities of the segment. Stress states outside this segment are not admitted in this concept (Fig.1).

- If the current stress at the structural iteration i, $\sigma^{(i)}$, remains on the segment the deformation variation is purely elastic. $f(\sigma^{(i)}, \kappa^{(i)}) < 0 \rightarrow \Delta\epsilon_e = \Delta\epsilon$
- If the current stress at the structural iteration i, $\sigma^{(i)}$, falls outside this segment $(f(\sigma^{(i)}, \kappa^{(i)}) \geq 0)$ the deformation is elastic–plastic. The proportion of elastic and plastic strain variations in Eq.(2) has to be determined. The stress $\sigma^{(i)}$ has to return simultaneously to the admissible domain.

*Note that the expression in Eq.(5) is more complex in a multi–dimensional case, since an equivalent stress measure depending on the invariants of the stress tensor is defined. This equivalent stress measure is then compared to the current yield stress $\sigma_h(\kappa^{(k)})$.

†Note that this means that the material exhibits the same behavior in tension and in compression.

1.3 The flow rule

The concept of a yield function has been introduced as the function that defines a section on the stress axis in 1D which separates the admissible from non–admissible stress states. Plastic deformation occurs if the stress point is on the limit of the yield segment and if it remains there. Plastic straining will occur only if the following two conditions are simultaneously met at state k:

$$\begin{cases} f(\sigma^{(k)},\, \kappa^{(k)}) = 0 \\ \dot{f}(\sigma^{(k)},\, \kappa^{(k)}) = \dfrac{\partial f}{\partial \sigma} d\sigma + \dfrac{\partial f}{\partial \kappa} d\kappa = 0 \end{cases} \tag{6}$$

with the second equation the so–called consistency requirement.
If plastic straining occurs, the variation in the plastic strain is determined by the flow rule, as the product of a scalar $\Delta\kappa$ and a vector quantity $\vec{n}$, following

$$\Delta\vec{\epsilon}_p^{(k)} = \Delta\kappa^{(k)}\, \vec{n}^{(k)} \tag{7}$$

In Eq.(7) $\Delta\kappa^{(k)}$ determines the magnitude of the plastic strain variation, while $\vec{n}^{(k)}$ describes the direction of the plastic flow. Both of these unknown quantities have to be determined during the solution of the plastic problem.

$$\vec{n}^{(k)} = \frac{\partial f(\sigma^{(k)},\, \kappa^{(k)})}{\partial \sigma} \tag{8}$$

In the 1D setting, considering equations (5) and (7) n is a scalar quantity.

1.4 Hardening behavior

Experimental results show that metallic materials show a *hardening behavior* with the accumulation of plastic strain. This hardening is related to the generation and interaction of dislocations. A phenomenological representation of this mechanism is the dependence of the yield function on a scalar measure of the plastic strain history, κ (as opposed to the perfect plastic behavior shown in Fig.1).

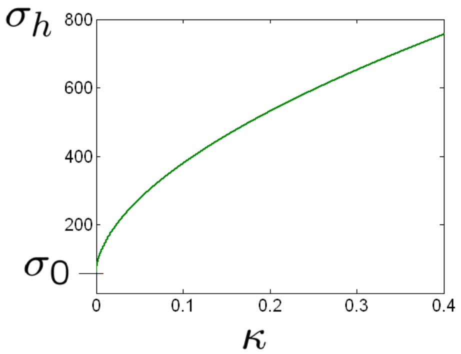

Figure 2: Hardening curve of type $\sigma_h = \sigma_0 + K\,\kappa^n$, with σ_0 [MPa] the initial yield limit.

This parameter is integrated along the loading path, and as in damage mechanics this history variable is monotonously increasing.

$$\kappa = \int \dot{\kappa}\; dt \tag{9}$$

Its value can never decrease, since it represents the cumulated plastic strain in the material. The yield function depends on the loading history only through a scalar–valued history parameter, the yield segment can only expand or shrink, but not translate on the stress axis. Because of this latter property this type of hardening is called isotropic hardening.

2 Numerical solution of the problem

The goal of this section is to show how an elastic–plastic problem is solved in the finite element method framework. The main difficulty in elastic–plastic computation is the computation of stresses, on which the internal forces and the global equilibrium of the system depend.

The equations of the mathematical theory of plasticity have to be solved to obtain the stress corresponding to the actual state of deformation. To obtain the stress corresponding to the actual load and to the constitutive law (which depends on the strain history) *an additional local iteration loop has to be inserted in the structural global one.* When the hardening behavior is nonlinear a Newton–Raphson iteration procedure is used in this local loop.

An extended class of solution algorithms is based on the decomposition of the incremental variation of the deformations (Eq.(2)). (This is the family of return mapping algorithms.) During the following discussion we will focus on the stress computation inside a finite element at a given structural iteration i. The incremental variation in the total strain, $\Delta\epsilon^{(i)}$ will be referred to without the superscript '(i)' to simplify the notations.

The stress computation itself is an iterative procedure. For the sake of clarity, superscripts '(k)' will refer to iterative values of the stress computation at the considered fixed structural iteration i. Converged values from the last increment are referred to with subscripts 'c' (e.g. σ_c and κ_c is the stress and the plastic strain history in the last converged equilibrium state, respectively). The stress increment at iteration k in such algorithms is given by:

$$\Delta\sigma^{(k)} = \mathrm{E}(\Delta\epsilon - \Delta\epsilon_p^{(k)}) \tag{10}$$

The update of the stresses is subdivided into two operations:

- Computation of the *elastic predictor*: σ_{tr} is obtained making the assumption that the total incremental strain variation is elastic $\Delta\epsilon_e = \Delta\epsilon$. If σ_{tr} is within the reversible domain, i.e. $f(\sigma_{tr}, \kappa_c) < 0$, the strain variation is purely elastic. The stress corresponds to the value of σ_{tr}.

- If the stress point obtained by the elastic predictor is outside of the yield segment, it is mapped back to the yield segment while the plastic deformation is increased. The increase in the plastic deformation results in the simultaneous expansion of the yield segment, if hardening is considered. This operation is called return mapping.

The nonlinear system of equations describing the plastic stress update can be summed up in the following way. The iterative stress update *from the last converged stress state*, σ_c, is done following

$$\sigma^{(k)} = \sigma_c + \mathrm{E}\,(\Delta\epsilon - \Delta\epsilon_p^{(k)}) \tag{11}$$

Introducing the elastic predictor, or *trial stress* as:

$$\sigma_{tr} = \sigma_c + \mathrm{E}\,\Delta\epsilon \tag{12}$$

Eq.11 can be expressed as:

$$\sigma^{(k)} = \sigma_{tr} - \mathrm{E}\,\Delta\epsilon_p^{(k)} \tag{13}$$

This can be rewritten in a residual form as:

$$\frac{\sigma^{(k)} - \sigma_{tr}}{\mathrm{E}} + \Delta\epsilon_p^{(k)} = 0 \tag{14}$$

Eq.(14) together with the condition that $f(\sigma^{(k)},\, \kappa^{(k)}) = 0$ compose a system of two equations to solve using a Newton–Raphson scheme.

$$\begin{cases} \dfrac{\sigma^{(k)} - \sigma_{tr}}{\mathrm{E}} + \Delta\epsilon_p^{(k)} = 0 \\ \\ \mathrm{abs}(\sigma^{(k)}) - \sigma_h(\kappa^{(k)}) = 0 \end{cases} \tag{15}$$

The Newton–Raphson procedure for stress and plastic strain history parameter update can be written as

$$\left\{ \begin{array}{c} \sigma^{(k+1)} \\ \\ \kappa^{(k+1)} \end{array} \right\} = \left\{ \begin{array}{c} \sigma^{(k)} \\ \\ \kappa^{(k)} \end{array} \right\} - [\mathbf{J}_p(\sigma^{(k)}, \kappa^{(k)})]^{-1} \left\{ \begin{array}{c} R^{(k)} \end{array} \right\} \tag{16}$$

where $\mathbf{J}_p$ is the Jacobian matrix of the system, and $R^{(k)}$ stands for the vector of residuals at the kth iteration of the stress update procedure (Eq.(15)).

$$\mathbf{J}_p(\sigma^{(k)}, \kappa^{(k)}) = \begin{bmatrix} \dfrac{\partial R_\epsilon}{\partial \sigma} & \dfrac{\partial R_\epsilon}{\partial \kappa} \\ \\ \dfrac{\partial R_f}{\partial \sigma} & \dfrac{\partial R_f}{\partial \kappa} \end{bmatrix} \tag{17}$$

with R_ϵ and R_f the first and second lines of the residuals in Eq.(15).
$\Delta\epsilon_p^{(k)}$ in Eq.(15) is computed based on Eq.(8) and

$$\Delta\kappa^{(k)} = \kappa^{(k)} - \kappa_c \tag{18}$$

Note that $\Delta\kappa^{(k)}$ is always positive, since the plastic strains are increasing during the stress update. The direction of the plastic strain n in Eq.(8) takes either 1 or -1 values, depending on the loading.
The upper left element of the inverse of the Jacobian matrix $\mathbf{J}_p^{-1}$ of the system in (Eq.(16)) is the *material tangent stiffness* $L = \dfrac{\partial \sigma}{\partial \epsilon}$, which is a scalar value in this 1D problem.