Geometrically nonlinear finite element modelling of linear elastic truss structrures

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1.5. Geometrically nonlinear bar finite element

Inspired and adapted from the ‘Nonlinear Modeling of Structures’ course of Prof. Thierry J. Massart at the ULB

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Derivation of F_{int} and K_{el}

Lab: Complete missing element relationships

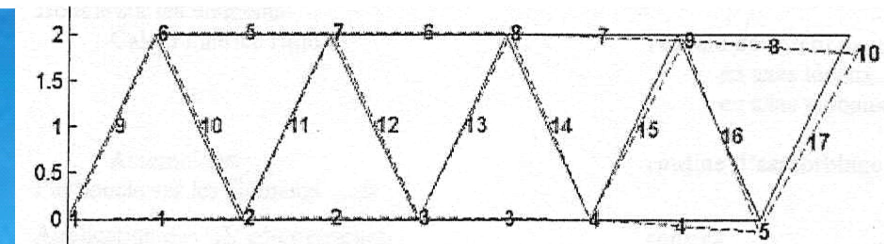
Analyse the 'V' shaped structure

ULB

Problem statement

Determine the displacements of structures at equilibrium

<u>Large</u> displacements and small deformations, linear elastic material



Stadium of Geneva

Nonlinear response?

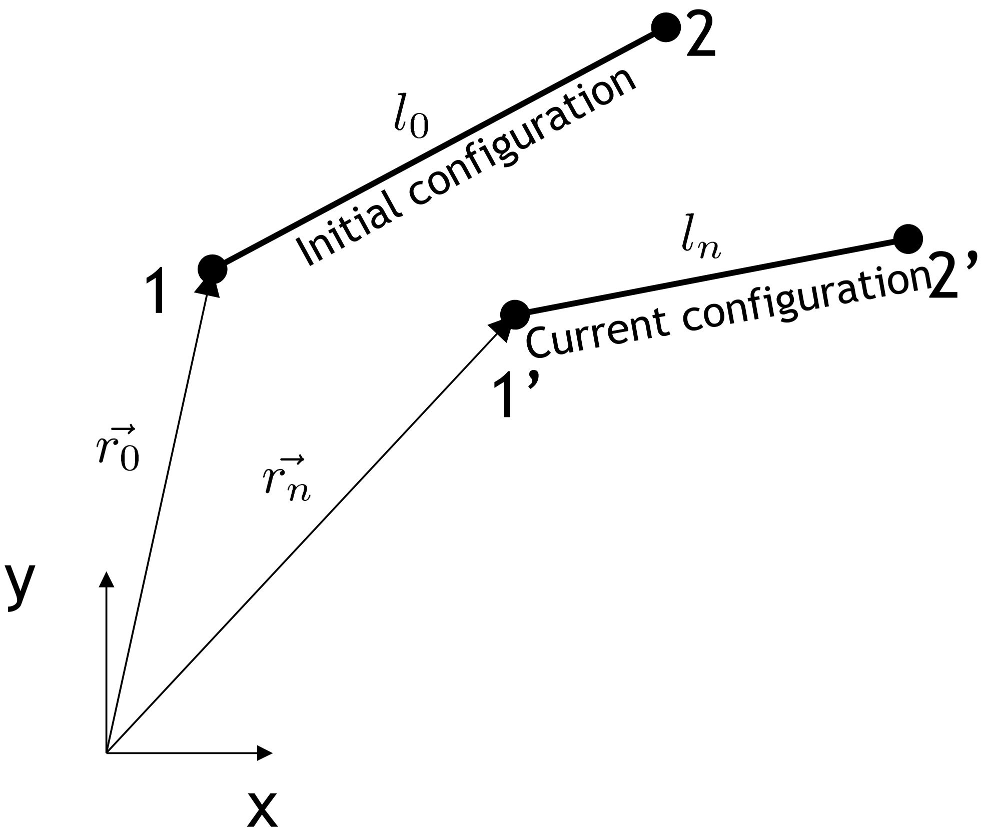
Linear elastic computation

Stadium of Geneva

Kuala Terengganu stadium, Malaysia, 2009

Nonlinear response

Kinematics of the bar



$$\vec{r}^{\,T} = \{\mathrm{x},\ \mathrm{y}\} \qquad \vec{u}^{\,T} = \{\mathrm{u},\ \mathrm{v}\}$$

$$\vec{r_n} = \vec{r_0} + \vec{u}$$

$$\mathbf{x}^T = \{x_1, x_2, y_1, y_2\}$$ nodal coordinates

$$\mathbf{u}^T = \underbrace{\{u_1, u_2, v_1, v_2\}}$$ nodal displacements

Order of dof!

Defining

$$\mathbf{x}_{21}^T = \{(x_2 - x_1),\ (y_2 - y_1)\}$$

$$\mathbf{u}_{21}^T = \{(u_2 - u_1),\ (v_2 - v_1)\}$$

Initial and current length of the bar

$$l_0^2 = 4\ \alpha_0^2 = \mathbf{x}_{21}^T\ \mathbf{x}_{21}$$

$$l_n^2 = 4\ \alpha_n^2 = (\mathbf{x}_{21} + \mathbf{u}_{21})^T\ (\mathbf{x}_{21} + \mathbf{u}_{21})$$

M. A. **Crisfield**, Non-linear Finite Element Analysis of Solids and Structures VOLUME 1: ESSENTIALS. John Wiley & Sons Ltd. Bafins Lane, Chichester West Sussex PO19 IUD, England, 1991, p65-70

Green strain measure

'Green' strain definition

$$\epsilon_G = \frac{l_n - l_0}{l_0} = \frac{(l_n - l_0)(l_n + l_0)}{l_0\ (l_n + l_0)} = \frac{l_n^2 - l_0^2}{l_0^2\ (2 + \epsilon_G)}$$ with ϵ_G small

$$\boxed{\epsilon_G = \frac{l_n^2 - l_0^2}{2 l_0^2} = \frac{(\mathbf{x}_{21} + \mathbf{u}_{21})^T\ (\mathbf{x}_{21} + \mathbf{u}_{21}) - \mathbf{x}_{21}^T \mathbf{x}_{21}}{2\ \mathbf{x}_{21}^T \mathbf{x}_{21}}}$$

because
$$\begin{cases} l_0^2 = 4\ \alpha_0^2 = \mathbf{x}_{21}^T\ \mathbf{x}_{21} \\ l_n^2 = 4\ \alpha_n^2 = (\mathbf{x}_{21} + \mathbf{u}_{21})^T\ (\mathbf{x}_{21} + \mathbf{u}_{21}) \end{cases}$$

with

$$\mathbf{x}_{21}^T = \{(x_2 - x_1),\ (y_2 - y_1)\}$$

$$\mathbf{u}_{21}^T = \{(u_2 - u_1),\ (v_2 - v_1)\}$$

Link between δε - δu and σ - ε

$$\epsilon_G = \frac{l_n^2 - l_0^2}{2l_0^2} = \frac{(\mathbf{x}_{21} + \mathbf{u}_{21})^T\ (\mathbf{x}_{21} + \mathbf{u}_{21}) - \mathbf{x}_{21}^T \mathbf{x}_{21}}{2\ \mathbf{x}_{21}^T \mathbf{x}_{21}}$$

$$\mathbf{b}_1^T = \frac{1}{4\alpha_0^2}\{-x_{21}, x_{21}, -y_{21}, y_{21}\}$$

$$\mathbf{b}_2(\mathbf{u})^T = \frac{1}{4\alpha_0^2}\{-u_{21}, u_{21}, -v_{21}, v_{21}\}$$

with $\begin{cases} x_{21} = x_2 - x_1 \\ y_{21} = y_2 - y_1 \\ \ldots \end{cases}$

Link between the variation of displacements and the variation of strain

$$\boxed{\delta\epsilon_G = \frac{\partial \epsilon_G}{\partial \mathbf{u}} \delta\mathbf{u} = (\mathbf{b}_1 + \mathbf{b}_2(\mathbf{u}))^T \delta\mathbf{u} = \mathbf{b}^T \delta\mathbf{u}}$$

Link between stress and strain

$$\boxed{\sigma_G = \epsilon_G\ E \quad \text{since linear elastic material}}$$

Expression of the internal force vector

Virtual work theorem $\delta \mathbf{u}_v$ is the virtual displacement vector

$$\sum_e \delta \mathbf{u}_v^T \; \mathbf{f}_{int} = \sum_e \int \sigma_G \delta \epsilon_v dV_0 = \sum_e \delta \mathbf{u}_v^T \int \sigma_G \mathbf{b} \; dV_0$$

because

$$\delta \epsilon_G = \frac{\partial \epsilon_G}{\partial \mathbf{u}} \delta \mathbf{u} = (\mathbf{b}_1 + \mathbf{b}_2(\mathbf{u}))^T \delta \mathbf{u} = \mathbf{b}^T \delta \mathbf{u}$$

$$\sigma_G = \epsilon_G \; E$$

Expression of the internal forces

$$\boxed{\mathbf{f}_{int} = \int \sigma_G \; \mathbf{b} \; dV_0 = 2\alpha_0 A_0 \sigma_G \; \mathbf{b}}$$

element cross-section

Derivation of the stiffness matrix (I/II)

Internal force vector

$$\mathbf{f}_{int} = \int \sigma_G \; \mathbf{b} \; dV_0 = 2\alpha_0 \; A_0 \; \mathbf{b} \; \sigma_G$$

Stiffness matrix of a bar

$$\mathbf{K}_t = \frac{\partial \mathbf{f}_{int}}{\partial \mathbf{u}} = 2\alpha_0 \; A_0 \; \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{u}} + 2\alpha_0 \; A_0 \frac{\partial \mathbf{b}}{\partial \mathbf{u}} \sigma_G$$

$$\mathbf{K}_t = \mathbf{K}_t^s + \mathbf{K}_t^g$$

Contribution of the stress change in a bar

$$\frac{\partial \sigma_G}{\partial \mathbf{u}} = E \frac{\partial \epsilon_G}{\partial \mathbf{u}} = E \; \mathbf{b}(\mathbf{u})^T$$

$$\mathbf{K}_t^s = 2\alpha_0 \; A_0 \; \mathbf{b} \frac{\partial \sigma_G}{\partial \mathbf{u}} = 2\alpha_0 \; A_0 \; E \; \mathbf{b}\mathbf{b}^T$$

Derivation of the stiffness matrix (II/II)

Contribution of the change in the geometry

$$\frac{\partial \mathbf{b}}{\partial \mathbf{u}} = \frac{\partial \mathbf{b}_2}{\partial \mathbf{u}} = \frac{1}{4\alpha_0^2}\,\mathbf{S} \qquad \mathbf{b_2}(\mathbf{u})^T = \frac{1}{4\alpha_0^2}(-u_{21}, u_{21}, -v_{21}, v_{21})$$

$$\mathbf{S} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{K}_t^g = 2\alpha_0\; A_0\, \frac{\partial \mathbf{b}}{\partial \mathbf{u}} \sigma_G = \frac{A_0 \sigma_G}{2\alpha_0} S$$

Expression of the stiffness matrix of a bar

$$\boxed{\mathbf{K}_t = 2\alpha_0\; A_0\; E\; \mathbf{b}\mathbf{b}^T + \frac{A_0 \sigma_G}{2\alpha_0} S}$$

Use of the element in the NL code

Incremental loop (for)

Initialize the residual

Iteration loop (while residual > tolerance)

Assembly of the tangent stiffness

Elimination of the prescribed and dependent dof

Solve the system

Substitute prescribed and dependent dof

Compute internal forces

Compute new residual

End of iteration loop

Save converged displacements

End of incremental loop

NL structural response