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Introduction to nonlinear finite element modeling

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3.0. Solution procedures

Inspired and adapted from the ‘Nonlinear Modeling of Structures’ course of Prof. Thierry J. Massart at the ULB

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Outline

Discretised equilibrium equation

Incremental-iterative procedure

Path-following methods (arc-length)

Based on

- → Non linear mechanics course, L.J. Sluys, R. de Borst, TU Delft, Netherlands
- → A course on Damage Mechanics, M. Geers, TU Eindhoven, Netherlands
- → Non linear finite elements, M. Crisfield, Tomes 1&2, Chichester, UK

Equilibrium problem - strong form

Volume translational equilibrium	$\sigma_{ij,j} + f_i = 0$
Rotational equilibrium	$\sigma_{[ij]} = 0$
Displacement boundary conditions	$u_i = \overline{u}_i$ sur S_u
Surface equilibrium	$\overline{T}_i^{(n)} = \sigma_{ij} n_j$ sur S_T
Strain-displacement relationship	$\text{defo}_{ij} = f_{NL}\left(u_{i,j}\right)$
Constitutive equations	$\sigma_{ij} = g_{NL}\,(\text{defo})$

Equilibrium problem: weak form

Among all the displacement fields u_i satisfying $u_i = \overline{u}_i$ on S_u

Find the one that satisfies

$$\underbrace{\int_v \sigma_{ij} \delta w_{(i,j)} dv}_{\delta W_{int}} = \underbrace{\int_v \delta w_i f_i dv + \int_{S_t} \delta w_i \overline{T}_i^{(n)} dv}_{\delta W_{ext}}$$

$\forall \delta w_i$ virtual such that $\delta w_i = 0$ on S_u

Finite element interpolation of the displacement

$$\left\{u^h(x)\right\} = [N(x)]\{q\} \qquad \left\{\delta w^h(x)\right\} = [N(x)]\{\delta d\}$$

$$\left\{\varepsilon^h\left(u^h\right)\right\} = [B]\{q\} \qquad \left\{\delta\varepsilon^h\left(w^h\right)\right\} = [B]\{\delta d\}$$

Discretised equilibrium $\forall\{\delta d\}$

$$\underbrace{\int_v [B]^t \left\{\sigma^h\left(u^h\right)\right\} dv}_{\{f_{int}\}} = \underbrace{\int_v [N]^t \{f\} dv + \int_{S_t} [N]^t \left\{\overline{T}\right\} ds}_{\{f_{ext}\}}$$

Discretised equilibrium

Non linear nature of the system $\{f_{int}\} = \{f_{ext}\}$

$$\{f_{int}\} = \sum_e \int_{V_e} [B]^T \{\sigma\} \, dV_e$$

$$\{f_{ext}\} = \sum_e \left(\int_{V_e} [N]^T \{f\} \, dV_e + \int_{S_{t,e}} [N]^T \left\{ \overline{T}^{(n)} \right\} dS_{t,e} \right)$$

Material non linearities

Non linear relation $\{\sigma\} = f_{nonlinear}(\{\varepsilon\})$

Geometrical non linearities

Choice of the reference configuration V_e (initial or deformed configuration)

The stress measure should be conjugated to the chosen strain measure

In the following only material non linearities will be considered for the sake of simplicity of notations

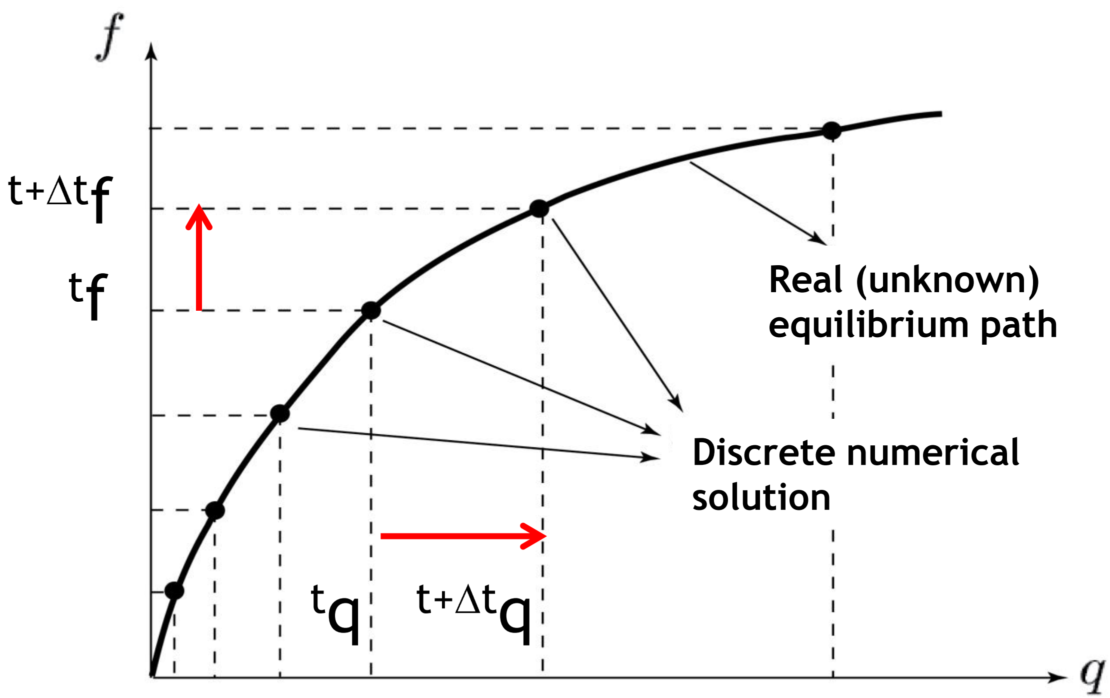
Why working with increments ?

The behaviour of materials may depend on the path followed

The radius of convergence of iterative procedures (NR) is limited

$\Rightarrow$ Apply loads with discrete, successive steps

$\Rightarrow$ The structural response is evaluated at discrete points



Incremental procedure

Definitions

We will use a 'sequencing' parameter t

- It allows defining the sequence of the different mechanical events
- This is a fictitious 'time' parameter (there is no material rate dependency)

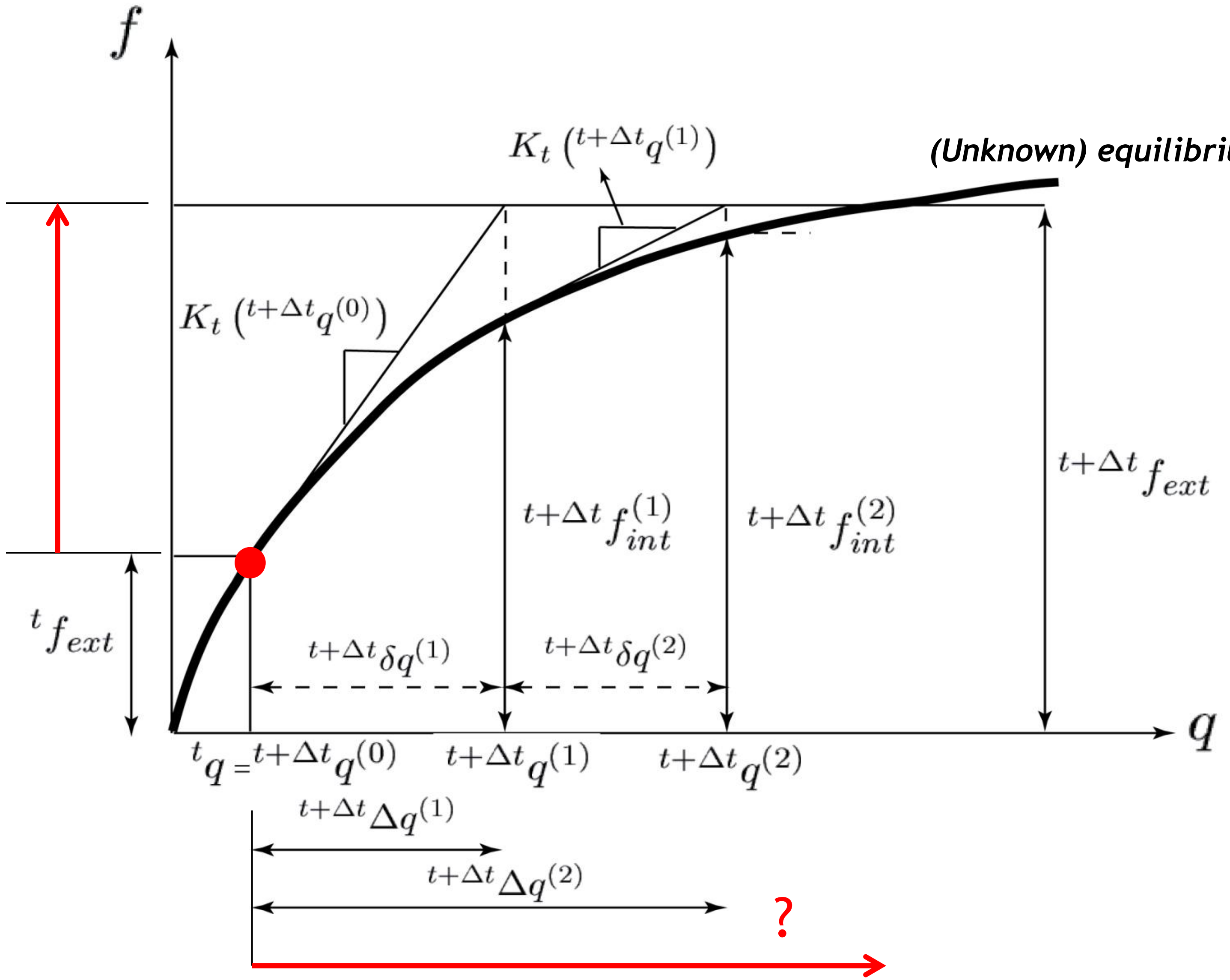
An increment or a step

- is defined as the set of operations needed to go from one state t to the next $t + \Delta t$ in an incremental solution procedure

Incremental update ΔX

- denotes the change in a quantity X during a full step or increment

Incremental-iterative procedure



Incremental-iterative procedure

Definitions

Iterative update ${}^{t+\Delta t}\delta X^{(i)}$

Change during the iteration *(i)* of the increment $t \rightarrow t + \Delta t$ of a mechanical quantity X

Incremental update at an iteration *(i)* ${}^{t+\Delta t}\Delta X^{(i)} = \sum_{(i)} {}^{t+\Delta t}\delta X^{(i)}$

Accumulation of all the iterative updates until iteration (i) during the increment $t \rightarrow t + \Delta t$ of a mechanical quantity X

Iterative corrections

Equilibrium at the end of iteration (i): one tries to impose

$$\underbrace{{}^{t+\Delta t}\{f_{int}\}^{(i)}}_{\text{nonlinear}} = {}^{t+\Delta t}\{f_{ext}\}$$

First order development around the approximation at iteration (i-1)

$${}^{t+\Delta t}\{f_{int}\}^{(i-1)} + \left[\frac{\partial\, {}^{t+\Delta t}\{f_{int}\}^{(i-1)}}{\partial q}\right] {}^{t+\Delta t}\{\delta q\}^{(i)} + \ldots = {}^{t+\Delta t}\{f_{ext}\}$$

$$\underbrace{\left[\frac{\partial\, {}^{t+\Delta t}\{f_{int}\}^{(i-1)}}{\partial q}\right]}_{[K_t({}^{t+\Delta t}\{q\}^{(i-1)})]} {}^{t+\Delta t}\{\delta q\}^{(i)} \simeq {}^{t+\Delta t}\{f_{ext}\} - {}^{t+\Delta t}\{f_{int}\}^{(i-1)}$$

New estimate of the incremental update

$${}^{t+\Delta t}\{q\}^{(i)} = {}^{t+\Delta t}\{q\}^{(0)} + {}^{t+\Delta t}\{\Delta q\}^{(i)} = {}^{t+\Delta t}\{q\}^{(0)} + \sum_{k=1}^{i} {}^{t+\Delta t}\{\delta q\}^{(k)}$$

Iterative corrections

Internal forces evaluation at the end of iteration (i)

$$ {}^{t+\Delta t}\{q\}^{(i)} \rightarrow {}^{t+\Delta t}\{\varepsilon\}^{(i)} \rightarrow {}^{t+\Delta t}\{\sigma\}^{(i)} $$

nonlinear

$$ \rightarrow {}^{t+\Delta t}\{f_{int}\}^{(i)} = \int_v [B]\ {}^{t+\Delta t}\{\sigma\}^{(i)}\, dv $$

Lack of equilibrium should be decreasing

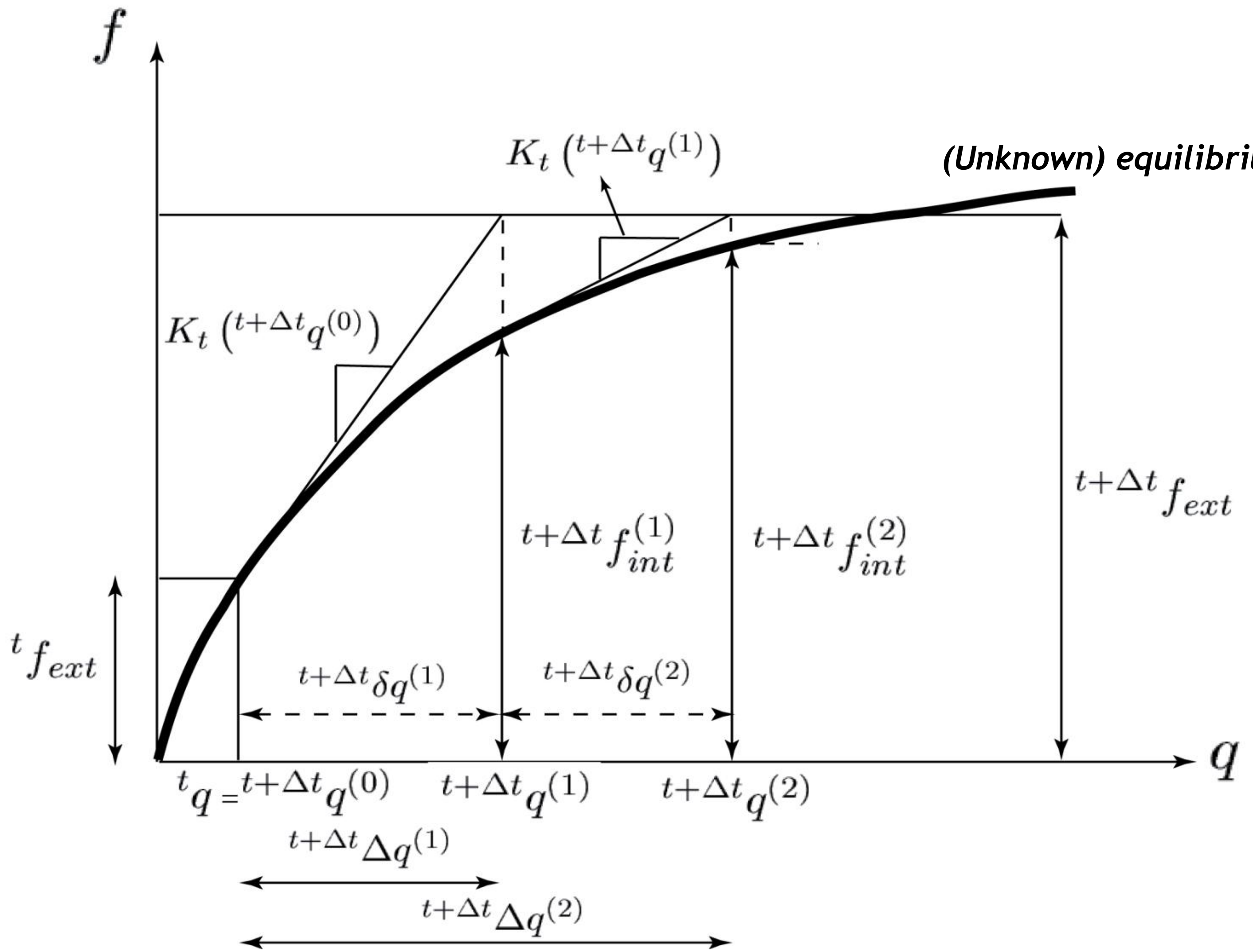
$$ {}^{t+\Delta t}\{f_{int}\}^{(i)} \neq {}^{t+\Delta t}\{f_{ext}\} $$

These successive developments have to be reapeated until lack of equilibrium vanishes

Remarks

- A different system matrix $\left[K_t\left({}^{t+\Delta t}\{q\}^{(i-1)}\right)\right]$ is inverted at each iteration
- The asymptotic convergence rate should be quadratic !

Uniaxial illustration



Global structure of non linear FE code

Loop on the loading (for)

Initialise the residual

Loop on iterations (while residual > tolerance)

Assembly of elementary stiffnesses

Eliminate prescribed dofs from the system

System solution $$[K_t]^{(i-1)}\left\{\delta q^{(i)}\right\} = \{f_{ext}\} - \left\{f_{int}^{(i-1)}\right\}$$

Substitution of prescribed dofs

Internal forces $$^{t+\Delta t}\{f_{int}\}^{(i)} = \int_v [B]\ ^{t+\Delta t}\{\sigma\}^{(i)}\, dv$$

Compute new residual

End of iteration loop

Archiving of results (stresses, strains, displacements, …)

End of loop on loading

Main control parameters

Loading type (prescribed forces or displacements)

Discretisation of the loading (phased, proportional)

Size of the loading steps (initial, maximum, minimum)

Definition of the convergence norm

Choice of a tolerance for convergence

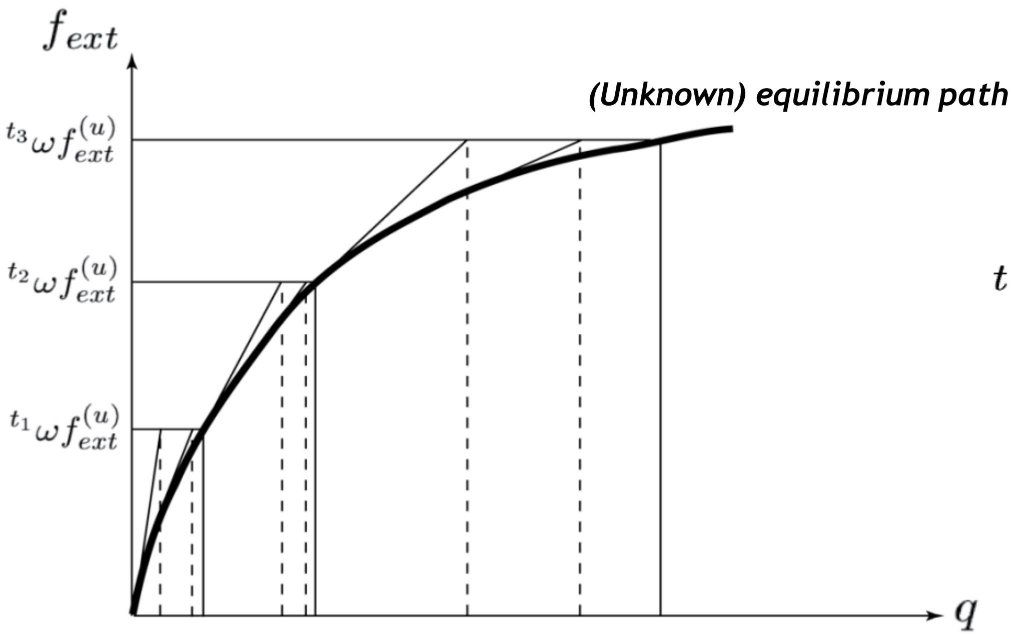
Step size refinement rules (when, how much, …)

Structural computation control

For each increment

- The procedure starts from an equilibrium point previously obtained
- An increase of the loading forces is applied
- An iterative procedure is used to obtain the equilibrium positions

If the loading is proportional, the load can be controlled by a scalar quantity and a unit force system $\left\{ f_{ext}^{(u)} \right\}$



$$ {}^{t}\{f_{ext}\} = {}^{t+\Delta t}\omega \left\{ f_{ext}^{(u)} \right\} $$

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Discrete response curve

This curve can be interpreted in (n+1) dimension space

The space $\{q_1, \ldots, q_n, \omega\}$ has n+1 dimensions

Geometrical interpretation of the response curve

Locus of points of the n+1 dimensions space where equilibrium is satisfied

Geometrical interpretation of the equilibrium problem

In the space $\{q_1, \ldots, q_n\}$ find the displacements such that

$$^{t+\Delta t}\{f_{int}\} = {}^{t+\Delta t}\{f_{ext}\}$$

In the space $\{q_1, \ldots, q_n, \omega\}$ find the intersection of the equilibrium curve and the surface of equation $\omega = {}^{t+\Delta t}\omega = C^{ste}$

$$^{t+\Delta t}\{f_{int}\} = \omega \left\{ f_{ext}^{(u)} \right\} \qquad \omega = {}^{t+\Delta t}\omega = C^{ste}$$

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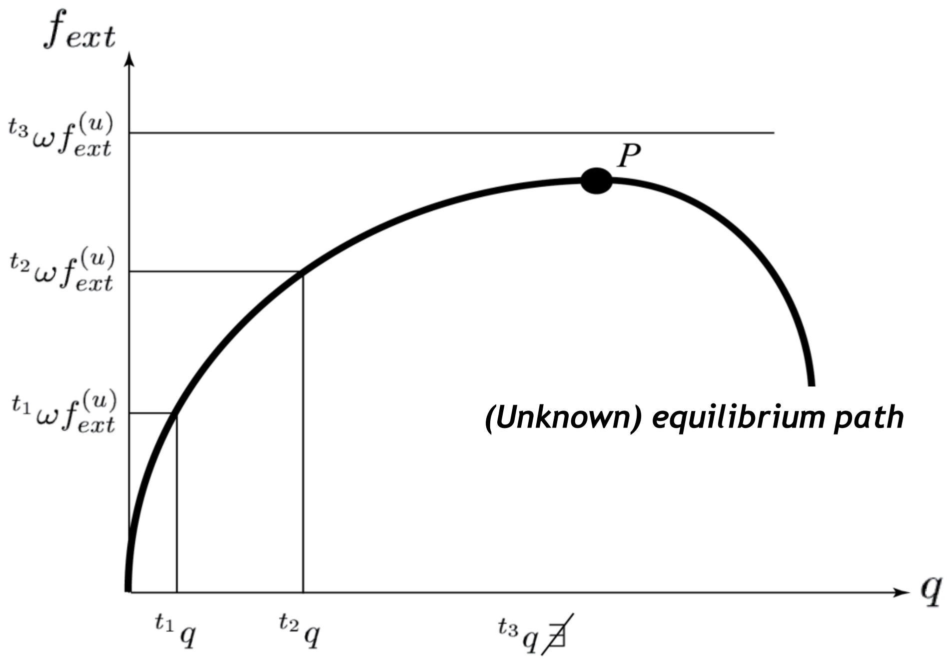
Why a need for an alternative control ?

Not always possible to find an intersection between the equilibrium curve and a hypersurface with increasing values of ω

Difficult to pass limit point for load control (like point P below)

This is linked to the non increasing variation of ω and to the choice of searching for an intersection of the equilibrium curve with $\omega = {}^{t+\Delta t}\omega = C^{ste}$



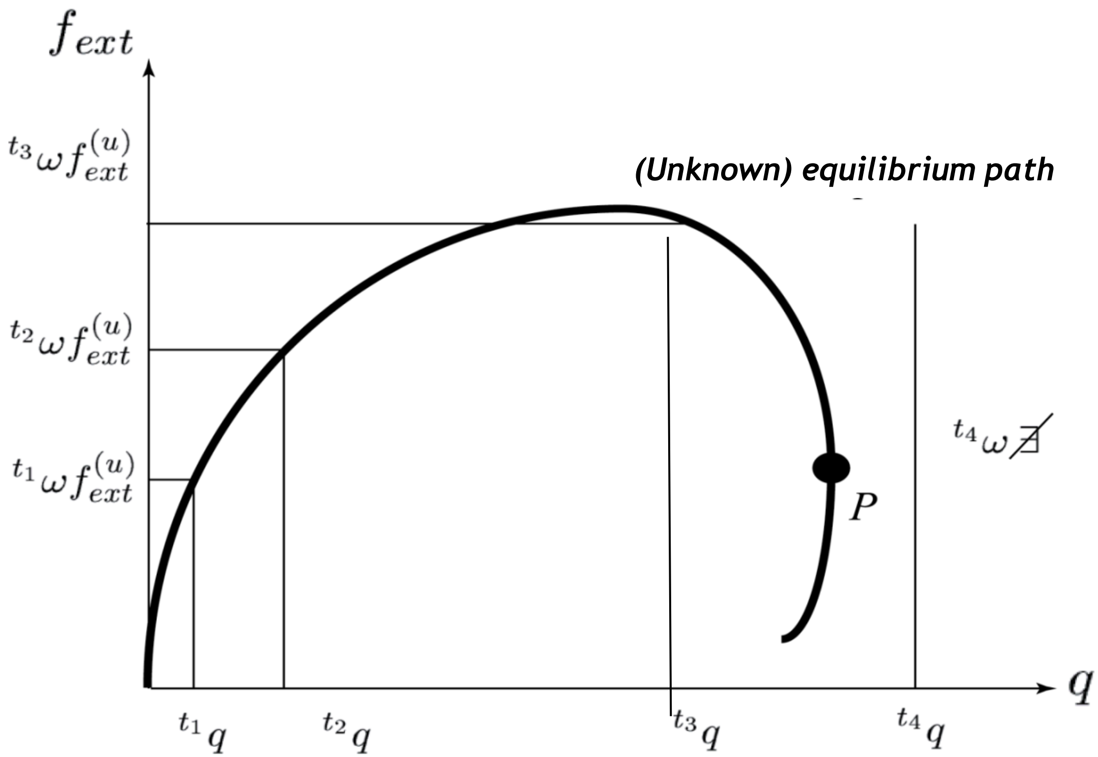
Try to find and intersection between the equilibrium curve and ANOTHER hypersurface controlled by A monotonically increasing quantity

Prescribed displacement control

Search for the intersection between equilibrium curve and a hyperplane of equation $q = {}^{t+\Delta t}q = C^{ste}$

Allows to pass load control limit points

But does not allow to pass displacement control limit points (point P)

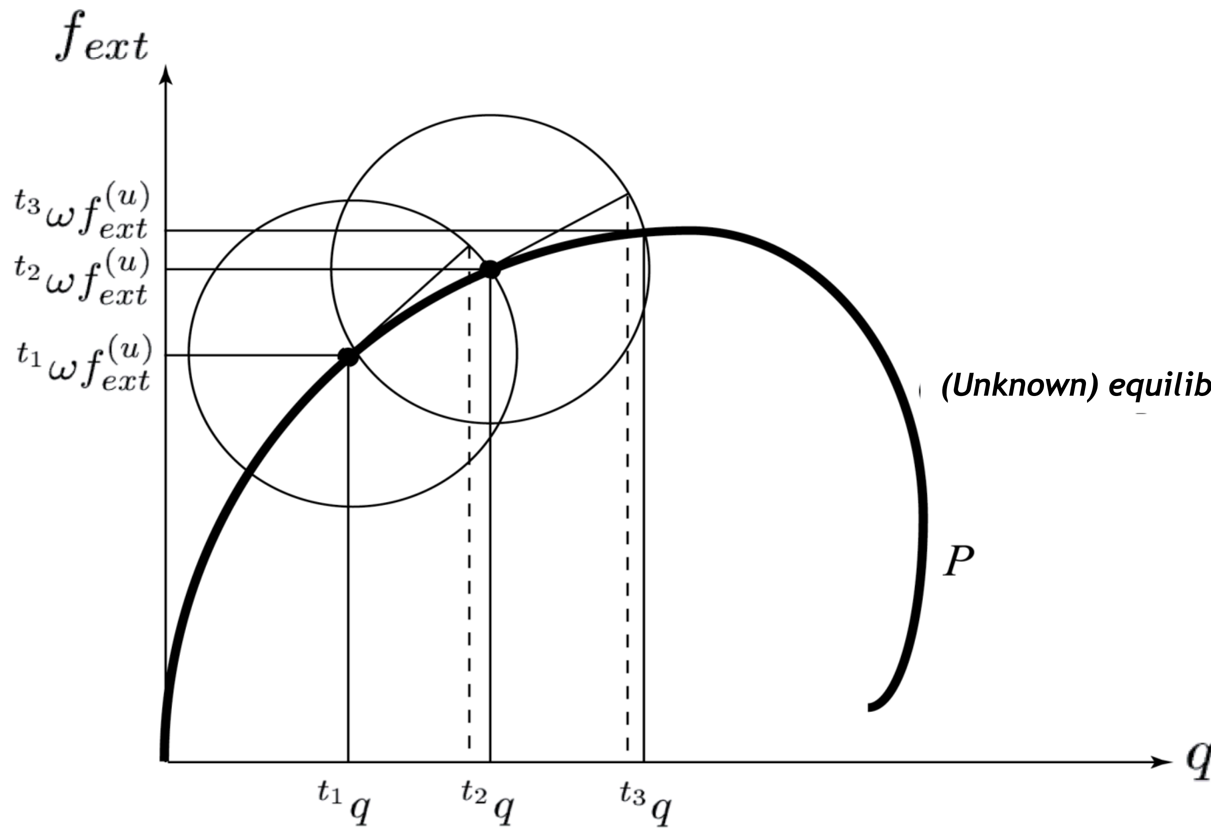


Arc-length control

Search for intersection of the curve with an hypercylinder or an hypersphere

Allows (theoretically) passing all types of limit points

BUT the load level ω is NOT constant during the iterations



Arc-length control

Make the load level ω variable between the successive iterations

Add a condition to determine this additional unknown

Equations to solve at iteration (i) of a given increment

Equilibrium $t \rightarrow t+\Delta t$

$$^{t+\Delta t}\{f_{int}\}^{(i)} = {}^{t+\Delta t}\omega^{(i)}\left\{f_{ext}^{(u)}\right\}$$

Which can be re-written in a residual form

$$\left\{r\left({}^{t+\Delta t}\{q\}^{(i)}, {}^{t+\Delta t}\omega^{(i)}\right)\right\} = {}^{t+\Delta t}\{f_{int}\}^{(i)} - \boxed{{}^{t+\Delta t}\omega^{(i)}}\left\{f_{ext}^{(u)}\right\} = 0$$

This is now an additional unknown

Arc-length control

An additional scalar equation is needed for this unknown

Write the equation of hypersurface centered on the point in state t

$$
{}^{t+\Delta t}\{\Delta q\}^{(i),T}\ {}^{t+\Delta t}\{\Delta q\}^{(i)} + \left({}^{t+\Delta t}\Delta\omega^{(i)}\right)^2 \Psi^2 \left\{f_{ext}^{(u)}\right\}^T \left\{f_{ext}^{(u)}\right\} = \Delta l^2
$$

Written under residual form

$$
{}^{t+\Delta t}a^{(i)} = {}^{t+\Delta t}\{\Delta q\}^{(i),T}\ {}^{t+\Delta t}\{\Delta q\}^{(i)} + \left({}^{t+\Delta t}\Delta\omega^{(i)}\right)^2 \Psi^2 \left\{f_{ext}^{(u)}\right\}^T \left\{f_{ext}^{(u)}\right\} - \Delta l^2 = 0
$$

Ψ = is a parameter of dimensional consistency

Ψ = 1 $\rightarrow$ spherical arc length

Ψ = 0 $\rightarrow$ cylindrical arc length

Arc-length control - implementation

Equilibrium at end of iteration (i): one tries to satisfy

$$\left\{r\left({}^{t+\Delta t}\{q\}^{(i)}, {}^{t+\Delta t}\omega^{(i)}\right)\right\} = {}^{t+\Delta t}\{f_{int}\}^{(i)} - {}^{t+\Delta t}\omega^{(i)}\left\{f_{ext}^{(u)}\right\} = 0$$

First order development of the residual around approx (i-1)

$$\begin{aligned} {}^{t+\Delta t}\{r\}^{(i)} = {}^{t+\Delta t}\{r\}^{(i-1)} &+ \left[\frac{\partial^{t+\Delta t}\{r\}^{(i-1)}}{\partial\{q\}}\right] {}^{t+\Delta t}\{\delta q\}^{(i)} \\ &+ \left\{\frac{\partial^{t+\Delta t}\{r\}^{(i-1)}}{\partial\omega}\right\} {}^{t+\Delta t}\delta\omega^{(i)} + \ldots \end{aligned}$$

$$\begin{aligned} {}^{t+\Delta t}\{r\}^{(i)} \simeq {}^{t+\Delta t}\{r\}^{(i-1)} &+ \left[K_t\left({}^{t+\Delta t}\{q\}^{(i-1)}\right)\right] {}^{t+\Delta t}\{\delta q\}^{(i)} \\ &+ \left\{f_{ext}^{(u)}\right\} {}^{t+\Delta t}\delta\omega^{(i)} \end{aligned}$$

Arc-length control - implementation

Impose on first order $\quad {}^{t+\Delta t}\{r\}^{(i)} = 0$

Iterative update of the displacements

$$ {}^{t+\Delta t}\{\delta q\}^{(i)} = {}^{t+\Delta t}\{\delta q_{nr}\}^{(i-1)} + \boxed{{}^{t+\Delta t}\delta\omega^{(i)}}\, {}^{t+\Delta t}\{q_e\}^{(i-1)} $$

$$ {}^{t+\Delta t}\{\delta q_{nr}\}^{(i-1)} = -\left[{}^{t+\Delta t}K_t^{(i-1)}\right]^{-1}\ {}^{t+\Delta t}\{r\}^{(i-1)} $$

with

$$ {}^{t+\Delta t}\{q_e\}^{(i-1)} = \left[{}^{t+\Delta t}K_t^{(i-1)}\right]^{-1}\ \left\{f_{ext}^{(u)}\right\} $$

Only ${}^{t+\Delta t}\delta\omega^{(i)}$ remains unknown in this iterative update of the displacement field if the (i -1) quantities are known

Arc-length control - implementation

Incremental updates of the unknowns

$$ {}^{t+\Delta t}\{\Delta q\}^{(i)} = {}^{t+\Delta t}\{\Delta q\}^{(i-1)} + {}^{t+\Delta t}\{\delta q_{nr}\}^{(i-1)} + \boxed{{}^{t+\Delta t}\delta\omega^{(i)}}\ {}^{t+\Delta t}\{q_e\}^{(i-1)} $$

$$ {}^{t+\Delta t}\Delta\omega^{(i)} = {}^{t+\Delta t}\Delta\omega^{(i-1)} + \boxed{{}^{t+\Delta t}\delta\omega^{(i)}} $$

Using these in the arc length expression

$$ {}^{t+\Delta t}\{\Delta q\}^{(i),T}\ {}^{t+\Delta t}\{\Delta q\}^{(i)} + \left({}^{t+\Delta t}\Delta\omega^{(i)}\right)^2 \Psi^2 \left\{f_{ext}^{(u)}\right\}^T \left\{f_{ext}^{(u)}\right\} = \Delta l^2 $$

This is scalar quadratic equation ${}^{t+\Delta t}\delta\omega^{(i)} \rightarrow {}^{t+\Delta t}\{\delta q\}^{(i)}$ is known

The remainder of the process is identical to a classical Newton scheme (stress and internal forces evaluation)

Summary

Predictor

Choose an estimate of the load increase $^{t+\Delta t}\Delta\omega^{(0)} = {}^{t+\Delta t}\delta\omega^{(0)}$

Compute the arc length $\Delta l = \sqrt{\left({}^{t+\Delta t}\Delta\omega^{(i)}\right)^2 \Psi^2 \left\{f_{ext}^{(u)}\right\}^T \left\{f_{ext}^{(u)}\right\}}$

Compute $^{t+\Delta t}\{\delta q_{nr}\}^{(0)}$, $^{t+\Delta t}\{q_e\}^{(0)}$

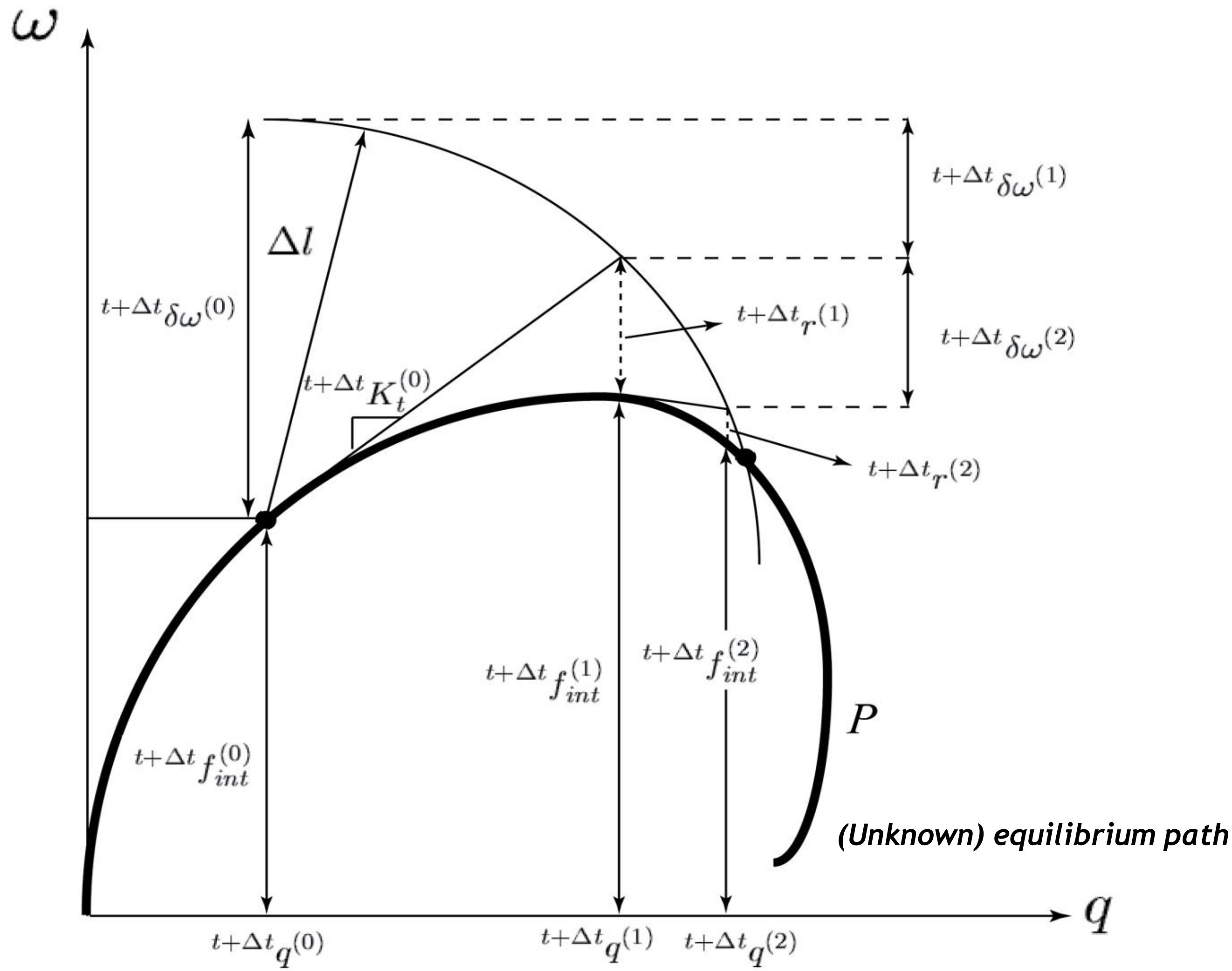
Compute $^{t+\Delta t}\delta\omega^{(1)}$ by solving the quadratic equation

Compute $^{t+\Delta t}\{\Delta q\}^{(1)}$ and $^{t+\Delta t}\{r\}^{(1)}$

Corrector

Repeat the last 3 stages of the predictor, increasing the iteration number of 1

Uniaxial illustration



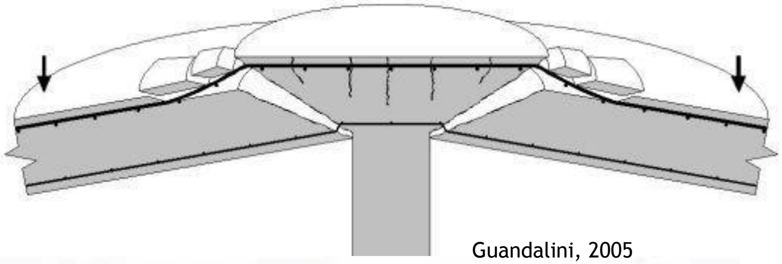
Remarks

Control parameters are similar to the load control case

Additional equations can be more complex than a sphere

- Linear equation for ${}^{t+\Delta t}\delta\omega^{(i)}$ (to avoid the presence of 2 roots)
- Use only one dof in the equation (if this is a monotonic quantity – often used in case of cracking)
- Change the controlling dof from step to step (crack mouth opening displacement control)

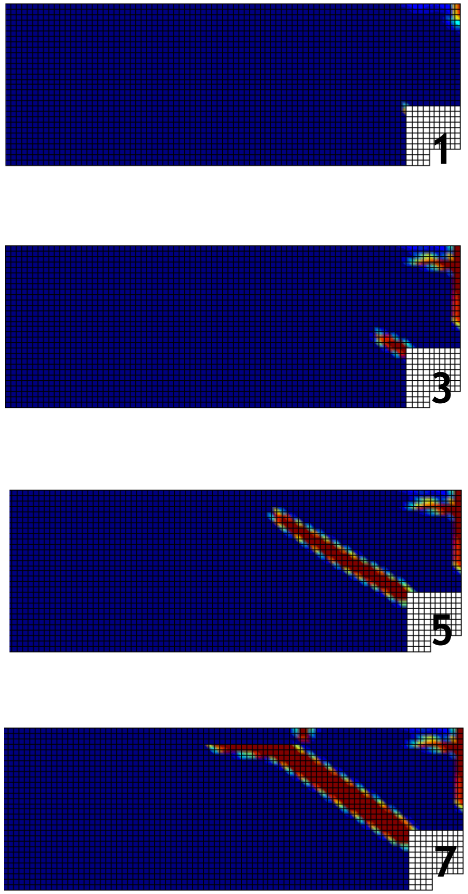
Structural failure (punching failure)



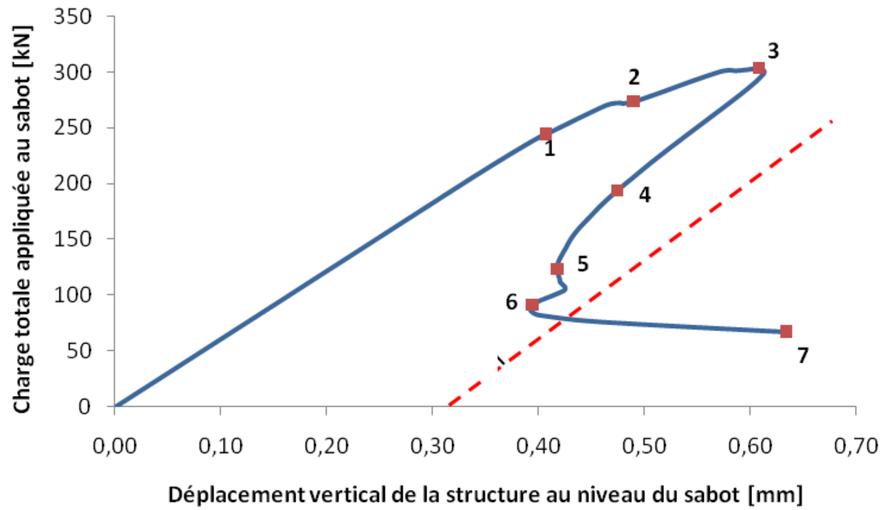
Guandalini, 2005

J.G.M. Wood, 2007

Punching of a plate - modelling

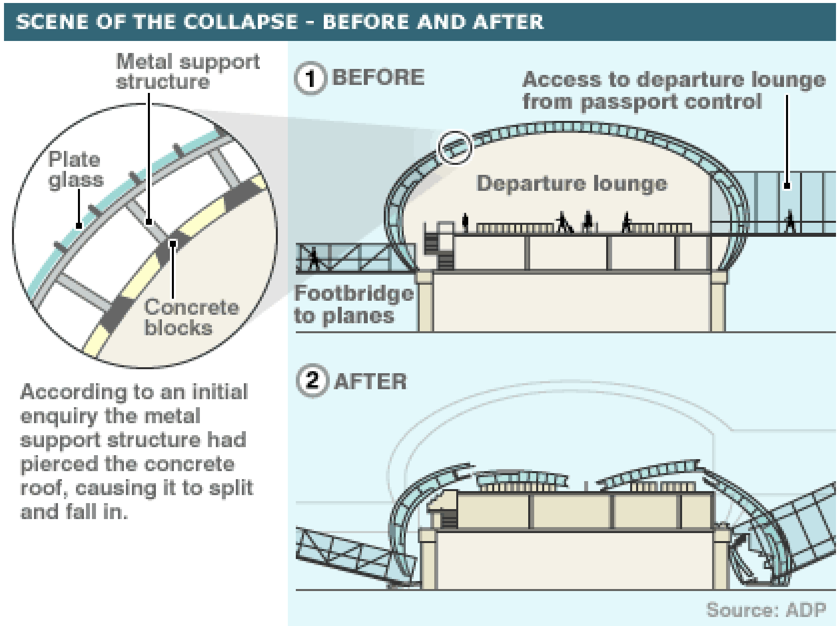


M. Syed Ali Azor, MFE, 2008



Punching failure

Charles de Gaulle Airport T2E
Cost ~ 750 M€, collapse 2004



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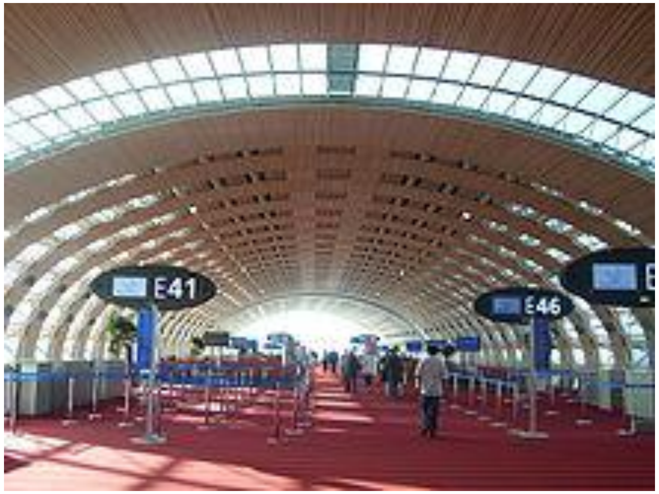


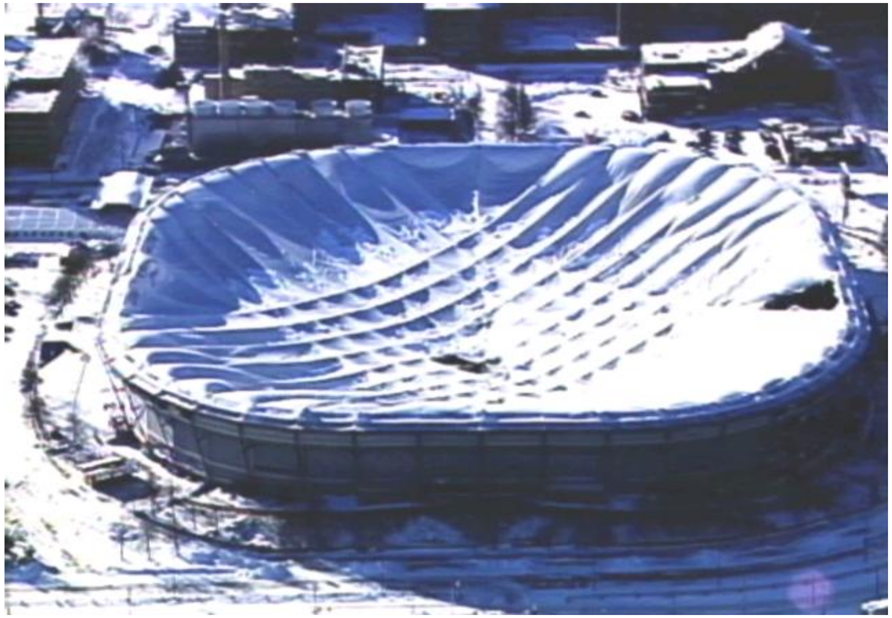
[http://en.wikipedia.org/wiki/File:Paris_Charles_De_Gaulle_Airport_Terminal_E_a.JPG]

[http://english.peopledaily.com.cn/200405/24/images/0524.paris2.jpg]

Instability

Metrodome, Minneapolis
December 2010

[http://www.themegaworldnews.com/wp-content/uploads/2010/12/dome_pic2.jpg]



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Applications